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Deformation of the Exterior Algebra $\Omega(M_n)$ and the Yang–Baxter Equation

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We show that the deformation of the exterior algebra on a given manifold is related to the existence of the Yang–Baxter equation. We prove that this deformed algebra involves a differential operator generating the algebra. The obtained differential calculus is not commutative and we recover the classical one for the classical limit of the deformation parameters. The q-analogue of the Leibniz rule corresponding to the purposed differential operator is given.

1. INTRODUCTION

The theory of the quantum group has gained much attention from physicists and mathematicians (Drinfel'd, 1985; Manin, 1989). This concept of deformation is based on the explicit dependence of these algebras on a given matrix satisfying the Yang–Baxter equation (YBE). It has been shown that the resolution of this equation allows the definition of a specific quantum algebra (or quantum group) having from the mathematical point of view the Hopf algebra structure. Another very large field of research in the same sense has been devoted to the introduction of noncommutative geometry using the solutions (Woronovicz, 1989) of the YBE. Many methods have been proposed to introduce a differential operator leading to a definition of a consistent differential calculus. The physical applications of these mathematical structures have led in several directions, for example, generalized statistics (anyons), low-dimensional condensed matter phenomena such as the fractional Hall effects (Lerda and Sciuto, 1993; Caracciolo and Monteiro, 1993; Daoud and Hassouni, 1996) and many other research fields.

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The present work is devoted to the deformation of the exterior algebra on a given differential manifold M_n. We prove that this deformation is related to the existence of the Yang–Baxter equations as for the other deformed algebras appearing in the literature. The appearance of this equation is of mathematical order; in fact, we show that its presence is ensured by the associativity of the deformed wedge product of the exterior algebra. As in the usual geometry, the deformed exterior algebra involves a differential operator mapping the pth-order form to the $(p + 1)$th-order one. By proposing some particular solutions of the YBE, we show that this differential calculus is not commutative. This coincides in some sense with the work (Woronovicz, 1989) in which a noncommutative differential calculus is introduced in another way. Indeed the space of the deformed one-forms can have the A-bimodule structure and the coordinates of a given point of M_n are seen to be the algebra A. We point out that this proposed noncommutative differential calculus recovers the classical one by taking the deformation parameters going to unity. We end this work by introducing the general form of the Leibniz rule corresponding to our differential operator d.

The paper is organized as follows: In Section 2 we give a brief review of the deformation of the exterior algebra on a given vector space. We present its relation with the YBE. In Section 3 we generalize this study to the case of a given manifold M_n. We introduce, using a particular solution of the YBE, a consistent differential operator. The final section is devoted to the introduction of the q-analogue of the Leibniz rule.

2. BRIEF REVIEW ON THE DEFORMATION OF $\Omega(V)$

Let V be a finite-dimensional vector space over $\mathbb{C}$ such that $\dim_{\mathbb{C}} V = n$ and $(e_i)_{i=1,\ldots,n}$ an arbitrary basis. We designate by $\sigma^i|_{i=1,\ldots,n}$ the basis of the dual space V^* of V, introduced to satisfy

$$\sigma^i(e_j) = \delta^i_j \tag{1}$$

The family $\sigma^i|_{i=1,\ldots,n}$ is nothing but an exterior one-form on V; the exterior product is then given by

$$\sigma^i \wedge \sigma^j = \sigma^i \otimes \sigma^j - \sigma^j \otimes \sigma^i \tag{2}$$

We define the deformation of $\Omega_1(V)$ (El Hassouni *et al.*, 1994) through a natural generalization of equation (2) as

$$\sigma^i \tilde{\wedge} \sigma^j = \sigma^i \otimes \sigma^j - \Lambda^{ij}_{kl}\sigma^k \otimes \sigma^l \tag{3}$$

with Λ^{ij}_{kl} the entries of some matrix $\Lambda \in \mathrm{End}_{\mathbb{C}}(V^* \otimes V^*)$:

$$\Lambda(\sigma^i \otimes \sigma^j) = \Lambda^{ij}_{kl}\sigma^k \otimes \sigma^l \tag{4}$$

The matrix Λ can be considered as a deformation candidate of the product $\wedge$ appearing in equation (2). Indeed, one can recover this equality (classical case) if we substitute Λ by the permutation matrix P ($P^{ij}_{kl} = \delta^i_l\delta^j_k$) in equation (3). This property allows us to imagine that Λ could depend on a given one or many complex parameters q's such that

$$\Lambda^{ij}_{kl}(q\text{'s} = 1) = \delta^i_l\delta^j_k \tag{5}$$

To construct the algebra of exterior forms built up from the deformed wedge products, we have assumed that the general expression of an arbitrary deformed two-form is given by

$$\omega^{(2)}_q = \omega_{ij}\sigma^i \tilde{\wedge} \sigma^j \tag{6}$$

We denote by $\{\sigma^i \tilde{\wedge} \sigma^j, i, j = 1, \ldots, n\}$ the set of the basis of vector space $\Omega^{(2)}_q(V)$ of all deformed two-forms constructed via the product $\tilde{\wedge}$.

It is obvious to remark that the antisymmetry property appearing for the wedge product in the classical limit ($\Lambda \to P$) is broken for $\Lambda \neq P$, i.e., $\sigma^i \tilde{\wedge} \sigma^j \neq -\sigma^j \tilde{\wedge} \sigma^i$.

Seeing that $\sigma^i \tilde{\wedge} \sigma^j$ belongs to $T(V)$ (the tensor algebra on V), its composition with another element of $\Omega_q(V)$ by $\tilde{\wedge}$ is introduced through the definition of the overlapping between the two operations $\tilde{\wedge}$ and $\otimes$ as

$$(\sigma^i \otimes \sigma^j) \tilde{\wedge} \sigma^k = \sigma^i \otimes \sigma^j \otimes \sigma^k - \Lambda^{jk}_{lm}\sigma^i \otimes \sigma^l \otimes \sigma^m + \Lambda^{jk}_{lm}\Lambda^{il}_{np}\sigma^n \otimes \sigma^p \otimes \sigma^m \tag{7a}$$

and

$$\sigma^i \tilde{\wedge} (\sigma^j \otimes \sigma^k) = \sigma^i \otimes \sigma^j \otimes \sigma^k - \Lambda^{ij}_{lm}\sigma^l \otimes \sigma^m \otimes \sigma^k + \Lambda^{ij}_{lm}\Lambda^{mk}_{np}\sigma^n \otimes \sigma^p \otimes \sigma^m \tag{7b}$$

The plus and minus signs appearing in these equalities are conventionally related to the number of Λ. In compact simple form (El Hassouni *et al.*, 1994) we rewrite equations (7) as follows:

$$(\sigma^1 \otimes \sigma^2) \tilde{\wedge} \sigma^3 = (E - \Lambda_{23} + \Lambda_{23}\Lambda_{12})\sigma^1 \otimes \sigma^2 \otimes \sigma^3 \tag{8a}$$

$$\sigma^1 \tilde{\wedge} (\sigma^2 \otimes \sigma^3) = (E - \Lambda_{12} + \Lambda_{23}\Lambda_{12})\sigma^1 \otimes \sigma^2 \otimes \sigma^3 \tag{8b}$$

where E is the identity matrix: $E^{ijk}_{lmn} = \delta^i_l\delta^j_m\delta^k_n$. We have

$$\Lambda_{23} = \mathbf{1} \otimes \Lambda, \qquad \Lambda_{12} = \Lambda \otimes \mathbf{1}$$

Now we write the deformed product of the three one-forms. By a straightforward computation, we get

$$(\sigma^1 \tilde{\wedge} \sigma^2) \tilde{\wedge} \sigma^3 = (E - \Lambda_{23} - \Lambda_{12} + \Lambda_{23}\Lambda_{12} + \Lambda_{12}\Lambda_{23} - \Lambda_{12}\Lambda_{23}\Lambda_{12})\sigma^1 \otimes \sigma^2 \otimes \sigma^3 \tag{9a}$$

$$\sigma^1 \tilde{\wedge} (\sigma^2 \tilde{\wedge} \sigma^3) = (E - \Lambda_{12} - \Lambda_{23} + \Lambda_{23}\Lambda_{12} + \Lambda_{12}\Lambda_{23} - \Lambda_{23}\Lambda_{12}\Lambda_{23})\sigma^1 \otimes \sigma^2 \otimes \sigma^3 \tag{9b}$$

The associativity of $\tilde{\wedge}$ is equivalent to the braid relation given by

$$\Lambda_{12}\Lambda_{23}\Lambda_{12} = \Lambda_{23}\Lambda_{12}\Lambda_{23} \tag{10}$$

An arbitrary deformed p-form is then expressed by

$$\omega_q^{(p)} = \omega_{i_1 \ldots i_p} \sigma^{i_1} \tilde{\wedge} \sigma^{i_2} \tilde{\wedge} \cdots \tilde{\wedge} \sigma^{i_p} \tag{11}$$

where $\omega_{i_1 \ldots i_p} \in \mathbb{C}$.

Now we are in position to give the definition of the deformed exterior product on V as

$$\Omega_\Lambda(V) = \bigoplus_{p \geq 0} \Omega_\Lambda^{(p)}(V), \qquad \Omega_\Lambda^{(0)}(V) = \mathbb{C} \tag{12}$$

$\Omega_\Lambda(V)$ is the space of the p-order deformed forms having the set $\{\sigma^{i_1} \tilde{\wedge} \cdots \tilde{\wedge} \sigma^{i_p}\}$ as a basis.

3. GENERALIZATION TO THE CASE OF A MANIFOLD M_n

In this section we will extend the previous construction to the case of a given finite-dimensional manifold M_n. As in the undeformed case, we introduce a differential calculus corresponding to the proposed deformation of the exterior algebra on M_n. We require d to satisfy the following:

(i) $d^2 = 0$.

(ii) d satisfies the graded Leibniz rule.

(iii) The differential calculus is invariant under transformations

$$x^i \rightarrow \alpha^i x^i$$

where x^i are the coordinates of a given point of M_n.

This condition will be very interesting when we give the deformation on $\Omega(M_n)$ below. One can remark that it is less restrictive than the invariance $GL_{q_{ij}}$ required in Brzezinski *et al.* (1992). The interpretation of this differential calculus in terms of the partial derivative of functions on M_n is given by

$$df(\cdots) = \sum_{i=1}^{n} \partial_i f(\cdots)\, dx^i \tag{13}$$

Following this scheme, we generalize equality (3) as

$$dx^i \tilde{\wedge} dx^j = dx^i \otimes dx^j - \Lambda_{kl}^{ij}\, dx^k \otimes dx^l \tag{14}$$

The associativity of $\tilde{\wedge}$ leads to the fact that Λ satisfies the braid equation (10). The construction of $\Omega_\Lambda(M_n)$ will then be based on the definition of d given by (13) and satisfying the above conditions. We point out that the deformed wedge product $\tilde{\wedge}$ can be seen as a generalization of the antisymmetric one $\wedge$. Indeed, the differential operator d in (13) recovers the classical one if one takes $\Lambda = P$ (P is the permutation operator).

The natural generalization of the classical wedge product $\wedge$ can then be expressed by

$$dx^i \tilde{\wedge} dx^j = -S^{ij}_{kl}\, dx^k \tilde{\wedge} dx^l \tag{15}$$

where S is a given $n^2 \times n^2$ matrix, $\tilde{\wedge} \to \wedge$ when $S \to P$.

Taking into account the relations (14)–(15), one has

$$(E_{12} - S_{12})(E_{12} + \Lambda_{12}) = 0 \tag{16}$$

Surprisingly enough, this equation coincides with the one containing the defining differential calculus in Brzezinski *et al.* (1992).

The most general matrix Λ satisfying (14) and (15) has the form

$$\Lambda = \sum_i e^i_i \otimes e^i_i + \sum_{i\neq j} q_{ij} e^i_j \otimes e^j_i \tag{17}$$

For the matrix S satisfying (16) and the braid relation ($\tilde{\wedge}$ is associative), one can show that it can be expressed as

$$S = \sum_i p_i e^i_i \otimes e^i_i + \sum_{i\neq j} q_{ij} e^i_j \otimes e^j_i \tag{18}$$

The construction of the deformed p from M_n will be given using the differential operator d. As for the case of vector space V [equation (12)], a general deformed p-form will be expressed by

$$\omega^{(p)}_\Lambda = \omega_{i_1 \ldots i_p}(x_1, \ldots, x_n) dx^{i_1} \tilde{\wedge} dx^{i_2} \tilde{\wedge} \cdots \tilde{\wedge} dx^{i_p} \tag{19}$$

where the coefficients $\omega_{i_1 \ldots i_p}$ are now functions on the variables $x_1, \ldots, x_n$. By employing the relation (13), we express a deformed $(p + 1)$-form by

$$d\omega^{(p)}_\Lambda = \partial_{i_0} \omega_{i_1 \ldots i_p}(x_1, \ldots, x_n)\, dx^{i_0} \tilde{\wedge} dx^{i_1} \tilde{\wedge} \cdots \tilde{\wedge} dx^{i_p} \tag{20}$$

At this step and by considering the proposed matrices Λ and S allowing the introduction of the exterior algebra $\Omega_\Lambda(M_n)$, we will discuss the consistency of d. In fact, the coordinates $x^i|_{i=1,\ldots,n}$ has been considered in the usual case $[\Omega_{\Lambda=P}(M_n)]$ as commuting variables (C-number). However, if one chooses such that $x^i x^j = x^j x^i$, the matrices Λ and S will reduce to P and the proposed differential calculus will be trivial. This point will be clarified when we will discuss the commutation relations between the variables and derivatives.

It has been proved that the solutions Λ and S [equations (17)–(18)] lead to the following commutation relations:

$$x - x: \qquad x^i \cdot x^j = q_{ij} x^j \cdot x^i$$

$$x - dx: \qquad x^i \cdot dx^i = p_i \, dx^i \cdot x^i$$

$$x^i \cdot dx^j = -q_j \, dx^j \cdot x^i \tag{21a}$$

$$dx - dx: \qquad dx^i \tilde{\wedge} dx^i = 0$$

$$dx^i \tilde{\wedge} dx^j = -q_{ij} \, dx^j \tilde{\wedge} dx^i \tag{21b}$$

$$\partial - x: \qquad \partial_i \cdot x^i = 1 + p_i x^i \cdot \partial_i, \qquad i \neq j$$

$$\partial_i \cdot x^j = \frac{1}{q_{ij}} x^j \cdot \partial_i \tag{21c}$$

$$\partial - dx: \qquad \partial_i \cdot dx^i = \frac{1}{p_i} dx^i \cdot \partial_i$$

$$\partial_i \cdot dx^j = \frac{1}{q_{ij}} dx^j \cdot \partial_i \tag{21d}$$

$$\partial - \partial: \qquad \partial_i \cdot \partial_j = q_{ij} \partial_j \cdot \partial_i \tag{21e}$$

In relations (21b) the product between the elements $dx - dx$ is the exterior deformed wedge product. From the algebraic point of view it can be considered as the dot $\cdot$ composing the other elements of the other equalities of (21). In what follows the operations $\tilde{\wedge}$ and $\cdot$ will be required to commute.

4. q-ANALOGUE OF THE LEIBNIZ RULE

We point out that the nilpotency condition on d is ensured by the equality $q_{ij} q_{ji} = 1$. Now, we present the analogue of the Leibniz rule for the exterior product,

$$d(\psi \wedge \eta) = d\psi \wedge \eta (-1)^{\deg \psi} \psi \wedge d\eta \tag{22}$$

where ψ and η are arbitrary forms in $\Omega(M_n)$.

In the case of the deformed wedge product this expression is not obvious; however, we will give an equivalent of it. As in Wess and Zumino (1990), the functions $f(x_1, \ldots, x_n)$ at a given point of M_n are viewed as polynomials on the coordinates. This form of the functions of M_n allows us to introduce the Leibniz rule corresponding to the deformed wedge product. Let us first discuss the case of two deformed one-forms ψ_Λ and η_Λ given by

$$\psi_\Lambda = \psi_{i_1}(x_1, \ldots, x_n)\, dx^{i_1} \tag{23a}$$

$$\eta_\Lambda = \eta_{j_1}(x_1, \ldots, x_n)\, dx^{j_1} \tag{23b}$$

The Leibniz rule corresponding to these forms is

$$d(\psi_\Lambda \tilde{\wedge} \eta_\Lambda) = d(\psi_{i_1}(x_1, \ldots, x_n)\, dx^{i_1} \tilde{\wedge} \eta_{j_1}(x_1, \ldots, x_n)\, dx^{j_1}) \tag{24}$$

where

$$\psi_{i_1}(x_1, \ldots, x_n) = \sum_{\mu_1, \ldots, \mu_n} a_{\mu_1 \ldots \mu_n} x_1^{\mu_1} \cdots x_n^{\mu_n}$$

and

$$\eta_{j_1}(x_1, \ldots, x_n) = \sum_{\nu_1, \ldots, \nu_n} b_{\nu_1 \ldots \nu_n} x_1^{\nu_1} \cdots x_n^{\nu_n} \tag{25}$$

By a direct computation using the equalities (21) and the relations

$$\begin{aligned} &\partial_{i_0}(x_1^{\mu_1} \cdots x_n^{\mu_n}) \\ &= \sum_{i=1}^{n-1} \delta_{i_0 i} q_{i_0 1}^{-\mu_1} q_{i_0 2}^{-\mu_2} \cdots q_{i_0 n}^{-\mu_n} q^{(1-\mu_i)/2} [\mu_i]_{q_{i_0 i}^{-1/2}} x_1^{\mu_1} \cdots x_i^{\mu_i - 1} \cdots x_n^{\mu_n} \\ &\quad + \prod_{i=1}^{n} q_{i_0 i}^{-\mu_i} x_i^{\mu_i} \partial_{i_0} \end{aligned} \tag{26}$$

one can then get

$$\begin{aligned} &d(\psi_\Lambda \tilde{\wedge} \eta_\Lambda) \\ &= d\left(\sum_{(\nu)(\mu)} a_{\mu_1 \ldots \mu_n} b_{\nu_1 \ldots \nu_n} x_1^{\mu_1} \cdots x_n^{\mu_n}\, dx^{i_1} \tilde{\wedge}\, x_1^{\nu_1} \cdots x_n^{\nu_n}\, dx^{j_1} \right) \\ &= \sum_{(\nu)(\mu)} a_{(\mu)} b_{(\nu)} \left\{ \prod_{l=1}^{n} q_{i_1 l}^{\nu_l}\, d(x_1^{\mu_1} \cdots x_n^{\mu_n} x_1^{\nu_1} \cdots x_n^{\nu_n}\, dx^{i_1} \tilde{\wedge}\, dx^{j_1} \right\} \\ &= \sum_{(\nu)(\mu)} a_{(\mu)} b_{(\nu)} \prod_{l=1}^{n} q_{i_1 l}^{\nu_l} \partial_{i_0} d(x_1^{\mu_1} \cdots x_n^{\mu_n} x_1^{\nu_1} \cdots x_n^{\nu_n})\, dx^{i_0} \tilde{\wedge}\, dx^{i_1} \tilde{\wedge}\, dx^{j_1} \\ &= \sum_{(\nu)(\mu)} a_{(\mu)} b_{(\nu)} \prod_{l=1}^{n} q_{i_1 l}^{\nu_l} \delta_{i_0} \left(\prod_{k'=1}^{n} q_{i_0 k'}^{-\mu_{k'}} \right) \\ &\quad \times q^{(1-\mu_k)/2} [\mu_k]_{q_{i_0 k}^{-1/2}} x_1^{\mu_1} \cdots x_i^{\mu_i - 1} \cdots x_n^{\mu_n} x_1^{\nu_1} \cdots x_i^{\nu_i - 1} \cdots x_n^{\nu_n} \\ &\quad + \prod_{k=1}^{n} q_{i_0 k}^{-\mu_k} x_k^{\mu_k} \partial_{i_0} (x_1^{\nu_1} \cdots x_n^{\nu_n})(dx^{i_0} \tilde{\wedge}\, dx^{i_1} \tilde{\wedge}\, dx^{j_1}) \end{aligned} \tag{27}$$

and finally, we have

$$d(\psi_\Lambda \tilde{\wedge} \eta_\Lambda) = \mathfrak{d}_1\psi(q_{ij}) \tilde{\wedge} \eta(q_{ij}) + (-1)^{\deg \psi}\psi(q_{ij}) \tilde{\wedge} \mathfrak{d}_2\eta(q_{ij})$$

where

$$\mathfrak{d}_1(\psi) = \sum_{(\mu)} \sum_{k=1}^{n-1} \delta_{i_0 k}\left(\prod_{k'=1}^{n} q_{i_0 k'}^{-\mu_{k'}}\right) q^{(1-\mu_k)/2}[\mu_k]_{q_{i_0 k}^{-1}/2} x_1^{\mu_1} \cdots x_i^{\mu_i - 1} \cdots x_n^{\mu_n}\, dx^{i_0} \tilde{\wedge} dx^{i_1}$$

$$\eta(q_{ij}) = \sum_{(\nu)} \prod_{l=1}^{n} q_{li_0}^{\nu_l} q_{li_1}^{\nu_l} b_{(\nu)} x_1^{\nu_1} \cdots x_i^{\nu_i - 1} \cdots x_n^{\nu_n}\, dx^{j_1}$$

$$\psi(q_{ij}) = q_{i_0 i_1} \sum_{(\mu)} a_{(\mu)} \prod_{k=1}^{n} q_{i_0 k}^{-\mu_{k'}} x_k^{\mu_k}$$

$$\mathfrak{d}_2\eta = \sum_{(\nu)} a_\nu \partial_{i_0}(x_1^{\nu_1} \cdots x_n^{\nu_n})\, dx^{i_0} \tilde{\wedge} dx^{j_1} \tag{28}$$

It is easy to see from (28) that one recovers the usual Leibniz rule for the classical limit of the deformation parameters.

This rule can be generalized to the case of two arbitrary forms ψ and η. The operators $\mathfrak{d}_1$ and $\mathfrak{d}_2$, formally introduced, reduce to the usual one, d.

5. CONCLUSION

In this paper we give a consistent deformation of the exterior algebra on M_n. The latter has been constructed starting from a definition of the differential operator allowing (as for the classical case) obtaining $\Omega_{q_{ij}}^p(M_n)$ for an arbitrary p. The introduction of d can be viewed as the construction of a noncommutative geometry (Manin, 1989) defined on a given quantum plane. In our context, the latter can be seen as the algebra A generated by the n coordinates $x_i|_{i=1,\ldots,n}$. We note also that this deformation led to the construction of an A-bimodule like the one encountered in Woronovicz (1989).

Elsewhere (Daoud *et al.*, n.d.) the deformation of the phase space (**x**, **P**) is constructed using these basic tools. This deformation allows us to construct a differential realization in the context of intermediate statistics seen as a natural generalization of supersymmetry using harmonic variables. We will discuss further the fractional spin notion by constraining the deformation parameters q_{ij}.

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